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Computation of lucky number of planar graphs is NP-hard

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ABSTRACT

A lucky labeling of a graph G is a function $\ell : V(G) \to \mathbb{N}$, such that for every two adjacent vertices v and u of G, $\sum_{w\sim v} \ell(w) \neq \sum_{w\sim u} \ell(w)$ ($x \sim y$ means that x is joined to y). A lucky number of G, denoted by $\eta(G)$, is the minimum number k such that G has a lucky labeling $\ell : V(G) \to \{1,\dots,k\}$. We prove that for a given planar 3-colorable graph G determining whether $\eta(G) = 2$ is **NP**-complete. Also for every $k \geqslant 2$, it is **NP**-complete to decide whether $\eta(G) = k$ for a given graph G.

1. Introduction

Graph coloring is one of the most studied subjects in graph theory. Recently, S. Czerwiński, J. Grytczuk and W. Zelazny [3] have studied the concept of lucky labeling as a vertex coloring. A labeling $\ell : V(G) \to \mathbb{N}$ is called lucky, if for every two adjacent vertices v and u of G, $\sum_{w\sim v} \ell(w) \neq \sum_{w\sim u} \ell(w)$ ($x \sim y$ means that x is joined to y). Lucky number of G is the minimum number k such that G has a lucky labeling $\ell : V(G) \to \mathbb{N}_k$, where $\mathbb{N}_k = \{1,\dots,k\}$. Those labelings arise as a vertex version of a well-known problem introduced by Karoński, Łuczak and Thomason [6]. They conjecture that every graph G except K_2 has an edge labeling in $\mathbb{N}_3$, such that assigning to each vertex of G the summation of the labels of its incident edges gives a proper vertex coloring of G. Also some other labelings have been studied extensively by several authors, for instance see [1,2,6,7]. In each of them, a labeling is an assignment of numbers to either the vertices or the edges or both of them. We consider only finite undirected simple graphs. Every graph has some lucky labeling, for example one may put the different powers of two $(1, 2, 4, \dots, 2^{|V(G)|-1})$ on the vertices of G.

A proper coloring of G is a function $c : V(G) \to \mathbb{N}$ such that for every two adjacent vertices v and u, we have $c(v) \neq c(u)$. A t-proper coloring is a proper coloring $c : V(G) \to \mathbb{N}_t$. G is called t-colorable if it has a proper coloring from the set $\mathbb{N}_t$. The smallest t such that G is t-colorable is called the chromatic number of G and it is denoted by $\chi(G)$. For a lucky labeling ℓ, define $f : V(G) \to \mathbb{N}$ as $f(v) = \sum_{w\sim v} \ell(w)$; so f is a proper coloring of G.

In this note we study the computational complexity of lucky number problem.

Theorem 1. *It is* **NP***-complete to decide for a given planar 3-colorable graph G, whether $\eta(G) = 2$.*

In [3] S. Czerwiński et al. have proved that the lucky number of planar graphs is bounded by a fix number, it has also been conjectured that for every graph G, $\eta(G) \leqslant \chi(G)$. Note that if this conjecture is true, then for graphs from Theorem 1, we have $2 \leqslant \eta(G) \leqslant 3$.

Our second theorem proves that the following problem for $k > 1$ is **NP**-complete.

Problem $\mathcal{L}_k$. Given a graph G, is $\eta(G) = k$?

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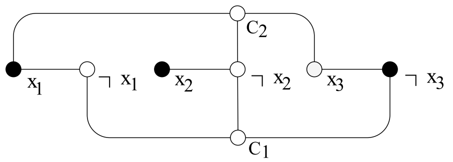

Fig. 1. The graph $G(\Phi)$ derived from the planar 3-SAT (type 2) formula $\Phi = c_1 \wedge c_2$, where $c_1 = \neg x_1 \vee \neg x_2 \vee \neg x_3$ and $c_2 = x_1 \vee \neg x_2 \vee x_3$. Φ is satisfied by the black vertices.

It is easy to see that $\mathcal{L}_1$ can be efficiently computed, in fact it is sufficient to check the degrees of every two adjacent vertices; $\eta(G) = 1$ if and only if no two adjacent vertices have the same degree.

Theorem 2. *For every $k \geqslant 2$, problem $\mathcal{L}_k$ is* **NP***-complete.*

For a graph G, we denote the degree of vertex v by either $d_G(v)$ or $d(v)$ and the maximum degree of G by $\Delta(G)$. We follow [8] for terminology and notation not defined here.

2. Proofs

Let Φ be a 3-SAT formula with clauses $C = \{c_1, \dots, c_k\}$ and variables $X = \{x_1, \dots, x_n\}$. Let $G(\Phi)$ be a graph with the vertices $C \cup X \cup (\neg X)$, where $\neg X = \{\neg x_1, \dots, \neg x_n\}$ such that for each clause $c_j = y \vee z \vee w$, c_j is adjacent to y, z and w; also every $x_i \in X$ is adjacent to $\neg x_i$. Φ is called a planar 3-SAT (type 2) formula if $G(\Phi)$ is a planar graph (see Fig. 1). It was shown that the problem of satisfiability of planar 3-SAT (type 2) is **NP**-complete [4] (is there a truth assignment for Φ that satisfies all the clauses?).

We reduce planar 3-SAT (type 2) problem to our problem. Consider an instance of planar 3-SAT (type 2) formula Φ with variables $V = \{x_1, \dots, x_n\}$ and clauses $C = \{c_1, \dots, c_k\}$. We transform this into a graph $G(\Phi)$ such that $\eta(G(\Phi)) = 2$ if and only if Φ is satisfiable.

We use two auxiliary graphs $B(x_j)$ and $A(c_j)$ which are shown in Figs. 2 and 3. The graph $G(\Phi)$ has a copy of $B(x_i)$ for each variable $x_i \in V$ and a copy of $A(c_j)$ for each clause $c_j \in C$. An edge $c_j x_i$ is added if c_j contains the literal x_i. See Fig. 4 for more details. Clearly $G(\Phi)$ is planar and $\chi(G(\Phi)) = 3$.

Assume that $\eta(G(\Phi)) \leqslant 2$ and $\ell : V(G(\Phi)) \to \{1, 2\}$ is a lucky labeling. We have the following two auxiliary lemmas.

Lemma 1. *For every variable x_j, we have $\ell(x_j) + \ell(\neg x_j) \geqslant 3$.*

Proof. Let x_j be an arbitrary variable, consider the subgraph $B(x_j)$. Since $f(y_j^2) \neq f(y_j^3)$ so $\ell(y_j^3) \neq \ell(y_j^2)$. Without loss of generality let $\ell(y_j^3) = 1$, thus $3 = f(y_j^1) \neq f(y_j^2) = 1 + \ell(y_j^1) + \ell(y_j^4)$ and $\ell(y_j^1) + \ell(y_j^4) \in \{3, 4\}$. If $\ell(y_j^1) + \ell(y_j^4) = 3$ then $f(y_j^3) = 5$ and if $\ell(y_j^1) + \ell(y_j^4) = 4$, then $f(y_j^2) = 5$. Therefore in both cases $5 \neq f(y_j^4) = 1 + 2 + \ell(x_j) + \ell(\neg x_j)$. $\square$

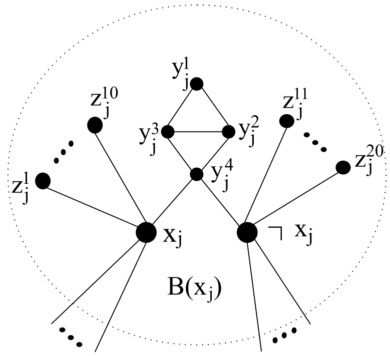

Fig. 2. The auxiliary graph $B(x_j)$.

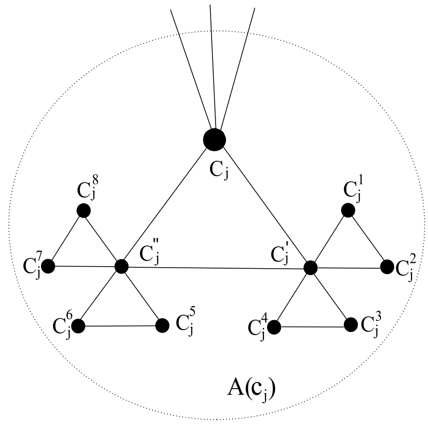

Fig. 3. The auxiliary graph $A(c_j)$.

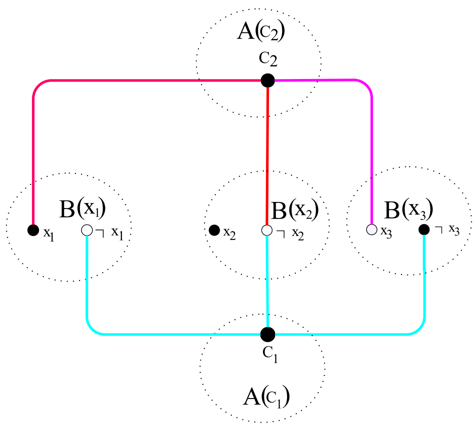

Fig. 4. The graph $G(\Phi)$ derived from Φ, explained in Fig. 1.

Lemma 2. *For every clause $c_j = a \vee b \vee c$, we have $\ell(a) + \ell(b) + \ell(c) < 6$.*

Proof. Suppose, by way of contradiction, that $c_j = a \vee b \vee c$ is a clause such that $\ell(a) + \ell(b) + \ell(c) = 6$. Clearly for $1 \leqslant i \leqslant 4$, $\ell(c_j^{2i-1}) + \ell(c_j^{2i}) = 3$. Therefore $f(c_j) = \ell(c_j') + \ell(c_j'') + 6$ and we have two similar equalities for $f(c_j')$ and $f(c_j'')$. Consequently ℓ is a lucky labeling for the odd cycle

$c_j c'_j c''_j$, but the lucky number of an odd cycle is 3. It is a contradiction. □

Proof of Theorem 1. First assume that $\eta(G(\Phi)) \leqslant 2$ and let $\ell : V(G(\Phi)) \to \{1, 2\}$ be a lucky labeling. Now we present a satisfying assignment $\Gamma : \{x_1, \ldots, x_n\} \to \{true, false\}$. By Lemma 1, for every x_i we have $\ell(x_i) + \ell(\neg x_i) \geqslant 3$ so it is impossible that both $\ell(x_i)$ and $\ell(\neg x_i)$ are 1. Now if $\ell(x_i) = 1$ let $\Gamma(x_i) = true$, if $\ell(\neg x_i) = 1$ let $\Gamma(x_i) = false$ and if $\ell(x_i) = \ell(\neg x_i) = 2$ consider an arbitrary assignment for $\Gamma(x_i)$. For every $c_j = a \vee b \vee c$ by Lemma 2, $\ell(a) + \ell(b) + \ell(c) < 6$; so at least one of the literals a, b, c is true. Consequently Γ satisfies c_j.

Next suppose that Φ is satisfiable with the satisfying assignment $\Gamma : \{x_1, \ldots, x_n\} \to \{true, false\}$. We present the lucky labeling ℓ for $G(\Phi)$ from the set $\{1, 2\}$. For each clause c_j let

$$\ell(c'_j) = \ell(c^1_j) = \ell(c^3_j) = \ell(c^5_j) = \ell(c^7_j) = 1,$$
$$\ell(c_j) = \ell(c''_j) = \ell(c^2_j) = \ell(c^4_j) = \ell(c^6_j) = \ell(c^8_j) = 2.$$

Also for every variable x_i, let $\ell(z^1_j) = \ell(z^2_j) = \cdots = \ell(z^{20}_j) = \ell(y^1_j) = \ell(y^3_j) = 1$ and $\ell(y^2_j) = \ell(y^4_j) = 2$. Moreover, if $\Gamma(x_i) = true$ set $\ell(x_i) = 1$ and $\ell(\neg x_i) = 2$, otherwise set $\ell(x_i) = 2$ and $\ell(\neg x_i) = 1$. Since Φ is satisfiable, by an easy counting one can see that ℓ is a lucky labeling. □

Corollary 1. *For every $k > 2$, it is* **NP***-complete to decide whether $\eta(G) = 2$ for a graph G with $\chi(G) = k$.*

Proof. It is easy to see that there is a graph H such that $\chi(H) = k$ and $\eta(H) = 1$. For example $V(H) = \{v_i \mid 1 \leqslant i \leqslant k\} \cup \{u_{ij} \mid 1 \leqslant j \leqslant i \leqslant k\}$, $E(H) = \{v_i v_{i'} \mid i \neq i'\} \cup \{v_i u_{ij} \mid 1 \leqslant j \leqslant i \leqslant k\}$. Now consider a new graph $G \cup H$ containing a copy of H and a copy of a given planar 3-colorable graph G such that two copies are disjoint. Then $\chi(G \cup H) = k$ and $\eta(G \cup H) = \eta(G)$. □

In order to prove Theorem 2, we reduce 3-Colorability to problem $\mathcal{L}_k$ for $k \geqslant 2$.

3-Colorability: Given a graph G; is $\chi(G) \leqslant 3$?

In [5], it has been shown that this problem is **NP**-complete.

Proof of Theorem 2. For a given graph G we construct a regular graph G^* such that $\chi(G^*) = \chi(G) + 6k - 8$ (step 1), next we make a graph G^{**} such that $\eta(G^{**}) = k$ if and only if $\chi(G^*) \leqslant 6k - 5$ (step 2). So $\eta(G^{**}) = k$ if and only if G is 3-colorable.

Step 1. For a given graph G consider a graph H containing a copy of G and a copy of the complete graph K_{6k-8} in such a way that every vertex of G is joined to every vertex of K_{6k-8}. We have $\Delta(H) = |V(G)| + 6k - 9$. For every vertex v of H join $\Delta(H) - d_H(v)$ new isolated vertices to v (therefore we have added $|V(G)|\Delta(H) - 2|E(G)|$ new vertices with degree one where $V(G)$ and $E(G)$ are the sets of vertices and edges). Next add $\lceil \frac{\Delta(H)-1}{2} \rceil$ copies of K_2's to this graph, call the resulting graph I, its vertices with

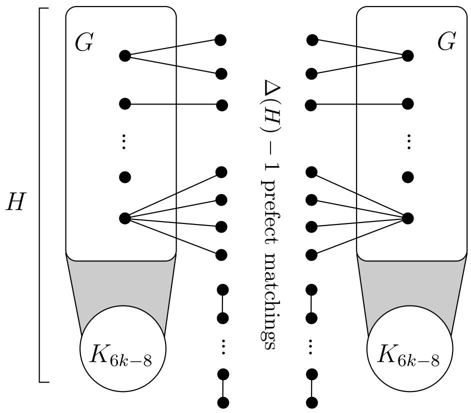

Fig. 5. The graph G^*.

degree $\Delta(H)$ old vertices and the vertices with degree 1 new vertices. Now consider two copies of I and $\Delta(H) - 1$ distinct prefect matchings between the new vertices of the one copy and the new vertices of the other copy. Note that indeed one can find $\Delta(H) - 1$ distinct perfect matchings in polynomial time. Now join two copies of I by those perfect matchings, name the constructed graph G^*. (Fig. 5.)

We study some properties of G^*. Clearly G^* is $\Delta(H)$-regular, also obviously $\chi(G^*) \geqslant \chi(G) + 6k - 8$. We show that the equality holds. Consider a proper $\chi(G)$-coloring for each of the two copies of G used in G^* by colors $\{1, \ldots, \chi(G)\}$ and color the vertices of the two copies of K_{6k-8}'s by colors $\{\chi(G) + 1, \ldots, \chi(G) + 6k - 8\}$. Also color the new vertices of one copy by $\chi(G) + 1$ and color the vertices of its K_2's by $\chi(G) + 1$ and $\chi(G) + 2$. For the second copy similarly use the colors $\chi(G) + 3$ and $\chi(G) + 4$. Note that since $k \geqslant 2$, then $6k - 8 \geqslant 4$ and therefore the colors $\chi(G) + 1, \ldots, \chi(G) + 4$ were used in coloring of K_{6k-8}'s previously. Consequently $\chi(G^*) = \chi(G) + 6k - 8$.

Step 2. Let n be the number of vertices of G^*. We can compute the value of n in polynomial time. Now, we construct G^{**}. First, consider a complete graph with vertices $X = \{x_{ij}\colon\ 0 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant k\}$, then add the independent vertices $\{y_s\colon\ 1 \leqslant s \leqslant n - 1\}$ to this graph and join x_{ij} to y_s if and only if $s \leqslant i \neq n$. Consider a copy of G^* with the vertices $Z = \{z_t\colon\ 1 \leqslant t \leqslant n\}$ and join every vertex of $\{x_{nj}\colon\ 1 \leqslant j \leqslant k\}$ to every z_t. Finally, join every z_t to six new vertices $z^1_t, \ldots, z^6_t$. Call the resulting graph G^{**}. (Fig. 6.)

We claim $\eta(G^{**}) = k$ if and only if $\chi(G^*) \leqslant 6k - 5$. First, note that in every lucky labeling ℓ of G^{**}, for every $1 \leqslant j_1 < j_2 \leqslant k$ we have $f(x_{nj_1}) \neq f(x_{nj_2})$, thus $\ell(x_{nj_2}) \neq \ell(x_{nj_1})$ (because all the neighbors of x_{nj_1} and x_{nj_2} are common except x_{nj_2} as a neighbor of x_{nj_1}, and vice versa). Therefore $\ell(x_{n1}), \ldots, \ell(x_{nk})$ are k distinct numbers (Fact 1), that means $\eta(G^{**}) \geqslant k$. Now suppose $\eta(G^{**}) = k$ and $\ell : V(G^{**}) \to \mathbb{N}_k$ is a lucky labeling of G^{**}. Let $\mathcal{F} = \{f(x)\colon\ x \in X\}$. Since the vertices of X form a complete graph, for every two vertices x and x' of X, $f(x) \neq f(x')$. Thus $\max_{x \in X} \mathcal{F} - \min_{x \in X} \mathcal{F} \geqslant |X| - 1 = k(n + 1) - 1$. On the other hand, let $M, m \in X$ be two vertices such that $f(M) = \max_{x \in X} \mathcal{F}$ and $f(m) = \min_{x \in X} \mathcal{F}$, we have:

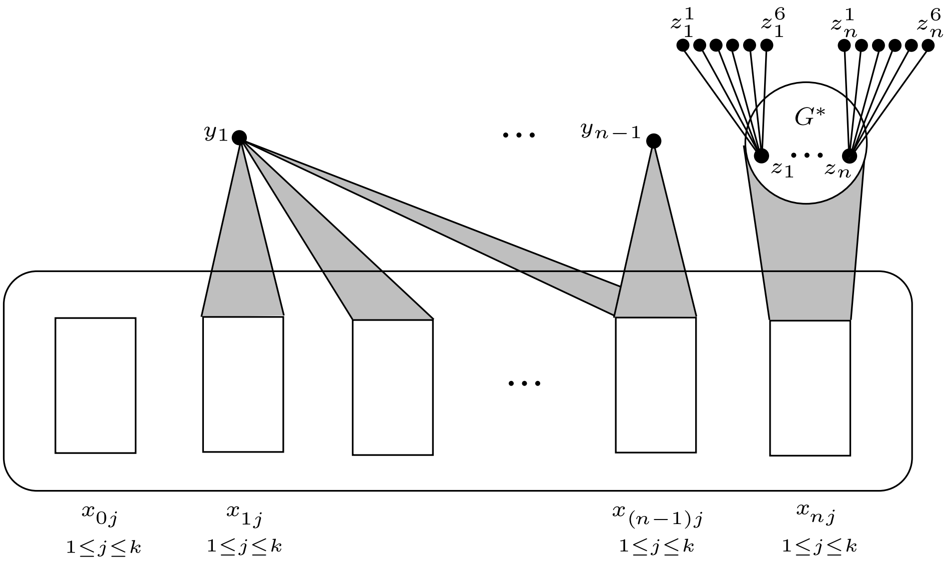

Fig. 6. The graph G^{**}.

$$k(n+1)-1 \leqslant f(M)-f(m)=\sum_{w\sim M}\ell(w)-\sum_{w\sim m}\ell(w) = \Bigl(\ell(m)+\sum_{\substack{w\in Y\cup Z\\ w\sim M}}\ell(w)\Bigr)-\Bigl(\ell(M)+\sum_{\substack{w\in Y\cup Z\\ w\sim m}}\ell(w)\Bigr).$$

The above inequality forces that the difference between $f(m)$ and $f(M)$ must be the maximum possible value, that is $\ell(m)=k$, $\ell(M)=1$, also m has no neighbor in $Y\cup Z$, M has n neighbors in $Y\cup Z$, and $\ell(z_s)=k$ for every s (Fact 2).

Now we study the luckiness of vertices of Z. For every s,

$$f(z_s)=\sum_{j=1}^{k}\ell(x_{nj})+\sum_{z_t\sim z_s}\ell(z_t)+\sum_{i=1}^{6}\ell\bigl(z_s^i\bigr).$$

By Facts 1 and 2:

$$f(z_s):=(1+2+\cdots+k)+k\Delta(H)+\sum_{i=1}^{6}\ell\bigl(z_s^i\bigr).$$

So the function $c(z_s):=\sum_{i=1}^{6}\ell(z_s^i)$ is a proper coloring of G^* with colors $\{6,7,\ldots,6k\}$, consequently $\chi(G^*)\leqslant 6k-5$.

On the other hand, let $\chi(G^*)\leqslant 6k-5$ and $c: Z\to\{6,7,\ldots,6k\}$ be a proper coloring of G^*. We present a lucky labeling for G^{**} with labels from $\mathbb{N}_k$, define

$$\ell(x_{ij})=j,\qquad \ell(y_s)=k,\qquad \ell(z_s)=k$$

and choose $\ell(z_s^1),\ldots,\ell(z_s^6)$ to be six arbitrary numbers from the set $\mathbb{N}_k$ such that $\sum_{i=1}^{6}\ell(z_s^i)$ is equal to $c(z_s)$. We have

$$f(x_{ij})=(n+1)\frac{k(k+1)}{2}-j+ki,$$

$$f(y_s)=(n-s)\frac{k(k+1)}{2},$$

$$f(z_s)=\frac{k(k+1)}{2}+k\Delta\bigl(G^*\bigr)+c(z_s),\qquad f\bigl(z_s^i\bigr)=k.$$

It is not hard to check that if $n\geqslant 5$ then ℓ is a lucky labeling of G^{**} with the labels from $\mathbb{N}_k$, so $\eta(G^{**})\leqslant k$. The proof is complete. □

Finally, we present the following conjecture:

Conjecture 1. *It is* **NP***-complete to decide whether* $\eta(G)=2$ *for a given* 3*-regular graph* G.

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